Output Feedback Stabilization for a Class of Multi-Variable Bilinear Stochastic Systems With Stochastic Coupling Attenuation

Qichun Zhang, Jinglin Zhou, Hong Wang, and Tianyou Chai

***Abstract*—In this technical note, stochastic coupling attenuation is investigated for a class of multi-variable bilinear stochastic systems and a novel output feedback m-block backstepping controller with linear estimator is designed, where gradient descent optimization is used to tune the design parameters of the controller. It has been shown that the trajectories of the closed-loop stochastic systems are bounded in probability sense and the stochastic coupling of the system outputs can be effectively attenuated by the proposed control algorithm. Moreover, the stability of the stochastic systems is analyzed and the effectiveness of the proposed method has been demonstrated using a simulated example.**

***Index Terms*—Bilinear stochastic systems, output feedback block backstepping, stochastic coupling attenuation.**

I. Introduction

Backstepping method [1], [2] has been regarded as a powerful tool for the nonlinear control system design which is formulated recursively in its design and analysis phase, where most of the existing results focus on the deterministic single-input single-output (SISO) control systems [3], [4]. For multiple-input multiple-ouptut (MIMO) control systems, global robust and adaptive control problems are presented in [5], [6] and [7], and a block backstepping design has been reported in [8] and [9]. Recently, the backstepping method has been extended to the control design for stochastic systems [10], [11] such as the backstepping design for high-order stochastic nonlinear systems with stochastic inverse dynamics [12]. Indeed, to apply backstepping control design methods to stochastic systems, there are some problems yet to be solved, such as how the closed-loop system outputs can be decoupled.

For MIMO stochastic systems, the coupling effect cannot be eliminated visually by the existing design methods because of the random noises. Motivated by statistical pairwise independence, the concept of output decoupling in the second moment sense is presented, where the outputs of the MIMO stochastic system can be considered as multi-dimensional measurable stochastic processes. If each two outputs of the MIMO stochastic system are pairwise independent, it means that the system is completely decoupled in the second moment sense. Practically, this concept means that the outputs of the closed-loop stochastic systems do not affect each other.

Since the pairwise independence can be described by covariance, the covariance matrix of the system outputs can be used as an ideal tool to analyze the stochastic couplings. If the covariance value equals to 0, it implies the complete decoupling in the second moment sense. Due to the accuracy of the stochastic models and the nonlinear effect, complete decoupling is difficult to achieve. Therefore, as a new concept stochastic coupling attenuation is proposed. The purpose of stochastic coupling attenuation is to reduce the mutual couplings of the closed-loop stochastic system outputs, which constitutes the physical meaning of this concept.

In this technical note, a novel m-block backstepping controller is designed by output feedback in order to stabilize the multi-variable bilinear stochastic systems. Meanwhile, to achieve output stochastic coupling attenuation in the second moment sense, the optimal parameters are obtained under the novel performance criterion which is based on covariance matrix.

The main contributions of the work are follows: (1) new concepts on stochastic decoupling and stochastic coupling attenuation for the closed-loop stochastic system are proposed; (2) a novel m-block backstepping design method is presented to stabilize the multi-variable bilinear stochastic systems using output feedback; (3) control parameter optimization under a novel criterion is established that minimizes the elements of the covariance matrix of the closed-loop system outputs.

II. Problem Formulation and Preliminaries

To describe the problem and the results, some mathematical notations will be used throughout this technical note. $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times m}$ stands for the real $n \times m$ matrix space; $Tr\{X\}$ denotes the trace of the square matrix X; $|X|$ denotes the Euclidean norm of a vector X and the corresponding induced norm for matrices is denoted by $\|X\|$; $\lambda_{\min}\{X\}$ and $\lambda_{\max}\{X\}$ represent the minimal eigenvalue and maximal eigenvalue of real matrix X, respectively. $\mathscr{C}^{1,2}$ denotes the set of all functions with continuous first and second partial derivatives; $\nabla_x f$ denotes the gradient of function f along x; $H_x\{f\}$ denotes the Hessian matrix of function f. $\mathscr{K}$ stands for the set of all functions which are continuous strictly increasing and vanish at zero; $\mathscr{K}_\infty$ denotes the set of all functions which are of class $\mathscr{K}$ and unbounded; $diag\{\cdot\}$ denotes the diagonal matrix.

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Q. Zhang is with the Control Systems Centre, School of Electrical Electronic and Engineering, The University of Manchester, Manchester, M13 9PL, U.K. (e-mail: qichun.zhang@manchester.ac.uk).

J. Zhou is with the College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, China (e-mail: jinglinzhou@mail.buct.edu.cn).

H. Wang is with the Pacific Northwest National Laboratory, Richland, WA 99352, USA (e-mail: hong.wang@pnnl.gov).

T. Chai is with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, China (e-mail: tychai@mail.neu.edu.cn).

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Consider a class of multi-variable bilinear stochastic systems with m blocks

$$\begin{aligned} d\bar{x}_1 &= (A_1\bar{x}_1 + \bar{x}_2)\,dt + \bar{x}_1 D_1 d\beta_t \\ &\vdots \\ d\bar{x}_i &= (A_i\bar{x}_i + \bar{x}_{i+1})\,dt + \bar{x}_1 D_i d\beta_t \\ d\bar{x}_m &= (A_m\bar{x}_m + \bar{u})\,dt + \bar{x}_1 D_m d\beta_t \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{1}$$

where β_t is the s-dimensional vector-valued Wiener process [13], $i = 1, 2, \ldots, m$ is the block index, $\bar{x}_i \in \mathbb{R}^n$ is the vector-valued system state for the i-th block, $A_i \in \mathbb{R}^{n\times n}$ is a real constant matrices and $D_i \in \mathbb{R}^s$ is real constant vector, $\bar{y} \in \mathbb{R}^n$ is the system output vector, $\bar{u} \in \mathbb{R}^n$ is the vector-valued control input. The underlying probability space is triple $(\Omega, \mathscr{F}, \mathrm{P})$, where Ω is the sample space of continuous functions, $\mathscr{F}$ is a filtration adapted to the Wiener process β_t, and P is the reference probability measure on Ω [10].

The noise term in the above equation is bilinear in terms of the output. This is a common phenomena in practical situation as it says that the sensor gain is subjected to a small drifting noise. Such a bilinear term can be expressed either in the state equation or in the output channel. Moreover, many industrial processes (such as nuclear, thermal, chemical processes, biology, socioeconomics, immunology, etc) can be approximately modeled in this form. For a further example, in the pH control process for papermaking systems the flow rate steam is coupled by the titration flow which can be expressed as a bilinear term for both the noise part and the main dynamic part as given in (1), see [14] and [15] for details.

System model (1) is in a block strict-feedback format where control inputs and system outputs are of the same dimension, and the outputs of such systems are dependent in the second moment sense. Therefore in this technical note our objectives are to design a control algorithm that can stabilize the closed-loop stochastic system and attenuate the output couplings in the second moment sense. The main results of this technical note will be shown in the following sections.

A. Output Coupling Attenuation in the Second Moment Sense

To formulate the control algorithm, the following definitions are introduced, where it will be seen that the stochastic decoupling control is defined in probability sense.

Definition 1: The MIMO stochastic system with n-dimensional outputs is said to be decoupled in the second moment sense, if for any positive constant $\varepsilon > 0$, the system outputs $y_1, \ldots y_n$ satisfy the following condition:

$$\lim_{t\to\infty} Pr\{\mathrm{cov}^2(y_i, y_j, t) \geq \varepsilon\}\,|_{1\leq i,\, j\leq n,\, i\neq j} = 0 \tag{2}$$

where $\mathrm{cov}(\cdot)$ denotes the covariance function. $Pr(\cdot)$ denotes the probability operator, $y = [y_1, \ldots, y_n]^T$ is the system output.

In practice, complete decoupling in the second moment sense given in Definition 1 is difficult to achieve. Therefore, we introduce the following definition on the stochastic coupling attenuation in the second moment sense.

Definition 2: The MIMO stochastic system with n-dimensional outputs is said to be coupling attenuation in the second moment sense, if the following cost function is minimized:

$$\min_{\bar{y}\in\Omega}\left(\lim_{t\to\infty} \mathrm{cov}^2(y_i, y_j, t)\,|_{1\leq i,\, j\leq n,\, i\neq j}\right) \tag{3}$$

In the following we transform the multi-objective optimization cost function (3) into a single-objective optimization criterion given by:

$$J_{obj} = \frac{1}{T}\int_0^T \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}^2(y_i, y_j, t)dt \tag{4}$$

where it can be seen that the minimum variance is guaranteed with $i = j$.

The stochastic couplings of the system outputs are attenuated if (4) is minimized. Moreover, the complete decoupling of the system outputs in the second moment sense is achieved if the criterion (4) converges to 0.

Remark 1: If the systems are subjected to the independent noises, the system outputs are stochastically decoupled once they are decoupled without the presence of the noise terms. This implies that stochastic decoupling is an extension of the decoupling control design for deterministic systems.

B. Bounded in Probability Sense

Consider the following stochastic system:

$$dx = f(x)\,dt + g(x)\,dw \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, w is an r-dimensional independent standard Wiener process [13], the underlying probability space is the triple $(\Omega, \mathscr{F}, \mathrm{P})$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitzian and satisfy

$$f(0) = 0, g(0) = 0 \tag{6}$$

Definition 3 ([16]): The solution process $\{x(t), t \geq 0\}$ of the stochastic system (5) is said to be bounded in probability if $\lim_{c\to 0}\sup_{0\leq t\leq\infty} P\{|x(t)| > c\} = 0$

Definition 4: For any given $V(x) \in \mathscr{C}^{1,2}$, associated with the stochastic differential equation (5), the differential operator $\mathscr{L}$ can be defined as follows:

$$\mathscr{L}V = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} Tr\left\{g^T(x)\frac{\partial^2 V}{\partial x^2} g(x)\right\} \tag{7}$$

We recall the following lemma [11] which gives the sufficient conditions on the boundedness in probability sense.

Lemma 1: Consider system (5) and suppose that there exists a positive-define and radially unbounded function $V(x) \in \mathscr{C}^{1,2}, \mu_1(\cdot), \mu_2(\cdot) \in \mathscr{K}_\infty$, positive-define and radially unbounded function $W(x)$ and constant $c > 0$ such that

$$\begin{aligned} \mu_1(|x|) &\leq V(x) \leq \mu_2(|x|) \\ \mathscr{L}V(x) &\leq -W(x) + c \end{aligned} \tag{8}$$

then the solution process of the system (5) is bounded in probability sense.

III. Output Feedback Stochastic Stabilization

In this section, linear estimator will be designed firstly because of the unmeasurable states of the system. Block backstepping technology is used to design a output feedback controller which makes the closed loop multi-variable bilinear stochastic system with linear estimator stabilized in probability sense.

A. Linear Estimator Design

The linear estimator is designed for unmeasurable states. The estimation error $\tilde{\bar{x}}_i = \bar{x}_i - \hat{\bar{x}}_i$ for the i th-block and $\tilde{x} = [\tilde{\bar{x}}_1, \ldots, \tilde{\bar{x}}_m]^T$

satisfy $d\tilde{x} = \tilde{A}\tilde{x}dt + \tilde{D}(\bar{y})\,d\beta_t$, where

$$\tilde{A} = \begin{bmatrix} A_1 - L_1 & I & & \\ -L_2 & A_2 & \ddots & \\ \vdots & \vdots & \ddots & I \\ -L_m & 0 & \dots & A_m \end{bmatrix}$$

$$\tilde{D}(\bar{y}) = diag\{\bar{y}, \dots, \bar{y}\}[D_1, \dots, D_m]^T = YD \tag{9}$$

Using filter or observer design approaches, $\tilde{A}$ can be designed to be Hurwitz and the multi-variable bilinear stochastic system with linear estimator can be expressed as

$$\begin{aligned} d\bar{y} &= \left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right)dt + \bar{y}D_1 d\beta_t \\ d\hat{\bar{x}}_2 &= \left(F_2 + \hat{\bar{x}}_3\right)dt \\ &\vdots \\ d\hat{\bar{x}}_i &= \left(F_i + \hat{\bar{x}}_{i+1}\right)dt \\ d\hat{\bar{x}}_m &= \left(F_m + \bar{u}\right)dt \\ d\tilde{x} &= \tilde{A}\tilde{x}dt + \tilde{D}(\bar{y})\,d\beta_t \end{aligned} \tag{10}$$

where $F_i := A_i\hat{\bar{x}}_i + L_i\left(\bar{y} - \hat{\bar{x}}_1\right), i = 2, \dots, m.$

Since the dynamic output equation and estimated state equations can be combined as a strict-feedback format, output feedback stabilization can be obtained by applying backstepping design procedure to the system $\left(\bar{y}, \hat{\bar{x}}_2, \dots, \hat{\bar{x}}_m, \tilde{x}\right)$.

B. Block Backstepping Controller Design

Note that the block backstepping design can be used for the stochastic system (10). For the first step, a new state transformation is introduced as follows:

$$\bar{z}_1 = \hat{\bar{x}}_2 - \bar{\phi}_1(\bar{y}) \tag{11}$$

where $\bar{z}_1 = [z_{11}, \dots, z_{1n}]^T$, function $\bar{\phi}_1(\bar{y}) : \mathbb{R}^n \to \mathbb{R}^n$ as the virtual control input which can be rewritten as $\bar{\phi}_1(\bar{y}) = [\phi_{11}(\bar{y}), \dots, \phi_{1n}(\bar{y})]^T$.

According to the well-known It ō's differentiation rule [17], the dynamics of the new transformed vector-valued state $\bar{z}_1$ is formulated as

$$\begin{aligned} d\bar{z}_1 &= d\hat{\bar{x}}_2 - d\bar{\phi}_1(\bar{y}) \\ &= \left(F_2 - \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 + \hat{\bar{x}}_3\right)dt \\ &\quad - \Phi_1\bar{y}D_1 d\beta_t \end{aligned} \tag{12}$$

where

$$\Phi_1 := [\nabla_{\bar{y}}\phi_{11}(\bar{y}), \dots, \nabla_{\bar{y}}\phi_{1n}(\bar{y})]^T$$

$$\Pi_1 := \begin{bmatrix} Tr\left\{(\bar{y}D_1)^T H_{\bar{y}}\{\phi_{11}(\bar{y})\}(\bar{y}D_1)\right\} \\ \vdots \\ Tr\left\{(\bar{y}D_1)^T H_{\bar{y}}\{\phi_{1n}(\bar{y})\}(\bar{y}D_1)\right\} \end{bmatrix} \tag{13}$$

To stabilize the state $\bar{z}_1$, a fourth-order Lyapunov function candidate is employed because of the It ō correction term.

$$V_1 = \frac{1}{4}\sum_{i=1}^{n} z_{1i}^4 \tag{14}$$

Using Definition 4, along the solution of equation (12), it can be obtained that

$$\begin{aligned} \mathscr{L}V_1 &= \frac{3}{2}Tr\left\{(\Phi_1\bar{y}D_1)^T\Gamma_1(\Phi_1\bar{y}D_1)\right\} \\ &\quad + \bar{\eta}_1\left(F_2 - \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 + \hat{\bar{x}}_3\right) \end{aligned} \tag{15}$$

where $\bar{\eta}_1$ and Γ_1 are used to simplify the expression of the formulation

$$\bar{\eta}_1 := \left[z_{11}^3, \dots, z_{1n}^3\right], \Gamma_1 := diag\left\{z_{11}^2, \dots, z_{1n}^2\right\} \tag{16}$$

A simple lemma is given below and it can be used repeatedly to simplify the first term of $\mathscr{L}V_1$ (15) which remains difficult to handle.

Lemma 2: Consider $A_1, A_2, B \in R^{n\times n}$ are n-dimensional square matrices and $D \in R^{n\times n}$ is diagonal matrix, where $A_1 = [\bar{a}_{11}, \dots, \bar{a}_{1n}]^T$, $A_2 = [\bar{a}_{21}, \dots, \bar{a}_{2n}]$ and $D = diag\{d_1, \dots, d_n\}$. Then the following inequality holds:

$$Tr\{DA_1BA_2\} \le \sum_{i=1}^{n}\|d_i\|\,\|\bar{a}_{1i}\|\,\|\bar{a}_{2i}\|\,\|B\| \tag{17}$$

Using Lemma 2 and Young's inequality [18], we can obtain the following result:

$$\begin{aligned} &Tr\left\{(\Phi_1\bar{y}D_1)^T\Gamma_1(\Phi_1\bar{y}D_1)\right\} \\ &= D_1D_1^T\,Tr\left\{\Gamma_1\Phi_1\bar{y}\bar{y}^T\Phi_1^T\right\} \\ &\le D_1D_1^T\sum_{i=1}^{n} z_{1i}^2\|\nabla_{\bar{y}}\phi_{1i}(\bar{y})\|^2\|\bar{y}\|^2 \\ &\le \sum_{i=1}^{n}\frac{\varepsilon_{1i}^2}{2}z_{1i}^4\|\nabla_{\bar{y}}\phi_{1i}(\bar{y})\|^4\|\bar{y}\|^4 + \sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\left\|D_1D_1^T\right\|^2 \end{aligned} \tag{18}$$

where ε_{1i} is a real positive constant for the i-th element.

Substituting the inequality (18) to $\mathscr{L}V_1$ (15), we can have

$$\begin{aligned} \mathscr{L}V_1 &\le \bar{\eta}_1\left(F_2 - \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 + \hat{\bar{x}}_3\right) \\ &\quad + \frac{3}{2}\sum_{i=1}^{n}\frac{\varepsilon_{1i}^2}{2}z_{1i}^4\|\nabla_{\bar{y}}\phi_{1i}(\bar{y})\|^4\|\bar{y}\|^4 + \frac{3}{2}\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\left\|D_1D_1^T\right\|^2 \\ &= \bar{\eta}_1\left(F_2 - \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 + \frac{3}{4}\Xi_1\|\bar{y}\|^4\right. \\ &\quad \left. + \hat{\bar{x}}_3\right) + \frac{3}{2}\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\left\|D_1D_1^T\right\|^2 \end{aligned} \tag{19}$$

where $\Xi_1 = \left[\varepsilon_{11}^2 z_{11}\|\nabla_{\bar{y}}\phi_{11}(\bar{y})\|^4, \dots, \varepsilon_{1n}^2 z_{1n}\|\nabla_{\bar{y}}\phi_{1n}(\bar{y})\|^4\right]^T$

Note that $\hat{\bar{x}}_3$ can be chosen to eliminate all the term of state $\bar{z}_1$. According to Definition 3 and Lemma 1, the transformed state $\bar{z}_1$ is bounded in probability sense with

$$\mathscr{L}V_1 = -\sum_{i=1}^{n}\lambda_{1i}z_{1i}^4 + c_1 \tag{20}$$

where λ_{1i} is a real positive constant for the i-th element and $c_1 = \sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\left\|D_1D_1^T\right\|^2 > 0$

The estimated state $\hat{\bar{x}}_3$ is a nonlinear function of estimation error $\tilde{x}$ and $\bar{y}$, the expression is given by

$$\begin{aligned} \hat{\bar{x}}_3 &= \Lambda_1\bar{z}_1 + \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) \\ &\quad + \frac{1}{2}\Pi_1 - \frac{3}{4}\Xi_1\|\bar{y}\|^4 - F_2 \end{aligned} \tag{21}$$

where $\Lambda_1 \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the diagonal element $-\lambda_{1i}$ and Λ_1 is negative-define matrix. Based on equation (21), we can simply define virtual control input $\bar{\phi}_2\left(\bar{y}, \hat{\bar{x}}_2\right) = \hat{\bar{x}}_3$ for iteration.

Similarly, the i-th step of the backstepping procedure can be given which starts from the new transformed state $\bar{z}_i$ for the i-th step

$$\bar{z}_i = \hat{\bar{x}}_{i+1} - \bar{\phi}_i\left(\bar{y}, \hat{x}_i\right) \tag{22}$$

where $\bar{z}_i = \left[z_{i1}, \ldots, z_{in}\right]^T$ and $\hat{x}_i^T = \left[\hat{\bar{x}}_2^T, \ldots, \hat{\bar{x}}_i^T\right]$, $i = 2, \ldots, m-1$.

Similar to equation (12), the dynamic of the transformed state $\bar{z}_i$ is given by

$$d\bar{z}_i = \left(F_{i+1} - \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 - \sum_{l=2}^{i}\Phi_l\left(F_l + \hat{\bar{x}}_{l+1}\right) + \hat{\bar{x}}_{i+2}\right)dt - \Phi_1\bar{y}D_1 d\beta_t \tag{23}$$

where $\Phi_l := \left[\nabla_{\hat{\bar{x}}_l}\phi_{l1}\left(\bar{y}, \hat{x}_l\right), \ldots, \nabla_{\hat{\bar{x}}_l}\phi_{ln}\left(\bar{y}, \hat{x}_l\right)\right]^T$.

Choose Lyapunov function candidate as $V_i = \frac{1}{4}\sum_{l=1}^{n} z_{il}^4$. Along the solution of equation (23), we have

$$\begin{aligned}
\mathscr{L}V_i &= \frac{3}{2}Tr\left\{\left(\Phi_1\bar{y}D_1\right)^T\Gamma_i\left(\Phi_1\bar{y}D_1\right)\right\} \\
&\quad + \bar{\eta}_i\left(-\Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - \frac{1}{2}\Pi_1 + F_{i+1} - \sum_{l=2}^{i}\Phi_l\left(F_l + \hat{\bar{x}}_{l+1}\right) + \hat{\bar{x}}_{i+2}\right) \\
&\le \bar{\eta}_i\left(-\Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) + \frac{3}{4}\Xi_i\|\bar{y}\|^4 + F_{i+1} + \hat{\bar{x}}_{i+2} - \sum_{l=2}^{i}\Phi_l\left(F_l + \hat{\bar{x}}_{l+1}\right) - \frac{1}{2}\Pi_1\right) \\
&\quad + \frac{3}{2}\sum_{l=1}^{n}\frac{1}{2\varepsilon_{il}^2}\left\|D_1 D_1^T\right\|^2
\end{aligned} \tag{24}$$

where ε_{il} is a real positive for the l-th element of the i-th step

$$\begin{aligned}
\bar{\eta}_i &:= \left[z_{i1}^3, \ldots, z_{in}^3\right] \\
\Gamma_i &:= diag\left\{z_{i1}^2, \ldots, z_{in}^2\right\} \\
\Phi_l &:= \left[\nabla_{\hat{\bar{x}}_l}\phi_{l1}\left(\bar{y}, \hat{x}_l\right), \ldots, \nabla_{\hat{\bar{x}}_l}\phi_{ln}\left(\bar{y}, \hat{x}_l\right)\right]^T \\
\Xi_i &:= \left[\varepsilon_{i1}^2 z_{i1}\|\nabla_{\bar{y}}\phi_{11}\left(\bar{y}\right)\|^4, \ldots, \varepsilon_{in}^2 z_{in}\|\nabla_{\bar{y}}\phi_{1n}\left(\bar{y}\right)\|^4\right]^T
\end{aligned} \tag{25}$$

To stabilize the state $\bar{z}_i$, the iterative state $\hat{\bar{x}}_{i+2}$ can be chosen as

$$\hat{\bar{x}}_{i+2} = \Lambda_i\bar{z}_i + \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - F_{i+1} + \frac{1}{2}\Pi_1 - \frac{3}{4}\Xi_i\|\bar{y}\|^4 + \sum_{l=2}^{i}\Phi_l\left(F_l + \hat{\bar{x}}_{l+1}\right) \tag{26}$$

where $\Lambda_i \in \mathbb{R}^{n \times n}$ is a diagonal negative-define matrix with the diagonal element $-\lambda_{ik}$, $k = 1, 2, \ldots, n$.

Notice that the control input $\bar{u}$ can be obtained when the index $i = m-1$ and the controller can be obtained by

$$\bar{u} = \Lambda_{m-1}\bar{z}_{m-1} + \Phi_1\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right) - F_m + \frac{1}{2}\Pi_1 - \frac{3}{4}\Xi_{m-1}\|\bar{y}\|^4 + \sum_{l=2}^{m-1}\Phi_l\left(F_l + \hat{\bar{x}}_{l+1}\right) \tag{27}$$

C. Stability Analysis in Probability Sense

Based on the structure of the controller (27), the closed-loop stochastic control system can be obtained and the following theorem states the main result on the stability for the closed-loop stochastic system.

Theorem 3: The multi-variable bilinear stochastic system with linear estimator (10) is bounded in probability sense using the controller (27) if there exist positive real constant b, δ_1 and δ_2, positive real constant set $\{\varepsilon_{0i} > 0 | i : 1 \to n\}$, diagonal positive-definite matrix P, W_0 and $\bar{\phi}_1\left(\bar{y}\right)$ which satisfy

$$\begin{aligned}
\bar{\phi}_1\left(\bar{y}\right) &\le -\left(W_0 + A_1\right)\bar{y} - \tilde{\bar{x}}_2 - \frac{3}{4}\left[\varepsilon_{01}^2 y_1, \ldots, \varepsilon_{0n}^2 y_n\right]^T\|\bar{y}\|^4 \\
& b\lambda_{\min}\{P\} - 8\delta_1^2 b^2 s - 2\delta_2^2 b^2 r > 0
\end{aligned} \tag{28}$$

where W_0 and P can be selected to satisfy the following conditions:

$$\begin{aligned}
&\lambda_{\min}\{W_0\} - \frac{1}{2\delta_1^2}\|D\|^4\|P\|^4 - \frac{1}{2\delta_2^2}\|D\|^4\|P\|^2\left|\lambda_{\max}\{P\}\right|^2 > 0 \\
&\tilde{A}^T P + P\tilde{A} = -I_1
\end{aligned} \tag{29}$$

where I_1 denotes the identity matrix with appropriate dimension.

Proof: Choosing Lyapunov function candidate for the closed-loop stochastic system (10) as

$$V = \frac{1}{4}\sum_{i=1}^{n} y_i^4 + \frac{b}{2}\left(\tilde{x}P\tilde{x}\right)^2 + \frac{1}{4}\sum_{i=1}^{m-1}\sum_{l=2}^{n} z_{il}^4 \tag{30}$$

then along the solution of the stochastic system (10), the infinitesimal generator is given by

$$\begin{aligned}
\mathscr{L}V &= \sum_{i=1}^{m-1}\mathscr{L}V_i + \left[y_1^3, \ldots, y_n^3\right]\left(A_1\bar{y} + \tilde{\bar{x}}_2 + \bar{\phi}_1\left(\bar{y}\right)\right) \\
&\quad + \frac{3D_1D_1^T}{2}Tr\left\{diag\left\{y_1^2, \ldots, y_n^2\right\}\bar{y}\bar{y}^T\right\} - b\tilde{x}^T P\tilde{x}\|\tilde{x}\|^2 \\
&\quad + 2bTr\left\{\tilde{D}^T\left(\bar{y}\right)\left(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P\right)\tilde{D}\left(\bar{y}\right)\right\}
\end{aligned} \tag{31}$$

Using the property of the matrix trace and Lemma 2, it can be obtained that

$$\begin{aligned}
&-b\tilde{x}^T P\tilde{x}\|\tilde{x}\|^2 \le -b\lambda_{\min}\{P\}\|\tilde{x}\|^4 \\
&\frac{3D_1D_1^T}{2}Tr\left\{diag\left\{y_1^2, \ldots, y_n^2\right\}\bar{y}\bar{y}^T\right\} \\
&\le \frac{3}{2}\sum_{i=1}^{n} y_i^2\|\bar{y}\|^2\left\|D_1D_1^T\right\| \\
&2bTr\left\{\tilde{D}^T\left(\bar{y}\right)\left(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P\right)\tilde{D}\left(\bar{y}\right)\right\} \\
&\le 4b\sqrt{s}\|P\|^2\|\bar{y}\|^2\|D\|^2\|\tilde{x}\|^2 \\
&+ 2b\sqrt{r}\|P\|\left|\lambda_{\max}\{P\}\right|\|\bar{y}\|^2\|D\|^2\|\tilde{x}\|^2
\end{aligned} \tag{32}$$

where $r = \max\{s, mn\}$ is a positive integer. In line with the structure of Y, we have $\|Y\| = \|\bar{y}\|$.

Using Young's inequality, it can be obtained that

$$
\begin{aligned}
&4b\sqrt{s}\|P\|^2\|\bar{y}\|^2\|D\|^2\|\tilde{x}\|^2\\
&\leq 8\delta_1^2 b^2 s\|\tilde{x}\|^4 + \frac{1}{2\delta_1^2}\|D\|^4\|P\|^4\|\bar{y}\|^4\\
&2b\sqrt{r}\|P\|\,|\lambda_{\max}\{P\}|\,\|\bar{y}\|^2\|D\|^2\|\tilde{x}\|^2 + (m-1)\,c_1\\
&\leq 2\delta_2^2 b^2 r\|\tilde{x}\|^4 + \frac{1}{2\delta_2^2}\|D\|^4\|P\|^2|\lambda_{\max}\{P\}|^2\|\bar{y}\|^4\\
&\frac{3}{2}\sum_{i=1}^{n} y_i^2\|\bar{y}\|^2\left\|D_1 D_1^T\right\|\\
&\leq \frac{3}{2}\sum_{i=1}^{n}\frac{\varepsilon_{0i}^2}{2}y_i^4\|\bar{y}\|^4 + \sum_{i=1}^{n}\frac{1}{2\varepsilon_{0i}^2}\left\|D_1 D_1^T\right\|^2
\end{aligned}
\tag{33}
$$

Substituting (28), (32) and (33) into (31), the infinitesimal generator can be finally represented as

$$
\begin{aligned}
\mathscr{L}V \leq &-\sum_{i=1}^{m-1}\sum_{l=1}^{n}\lambda_{il}z_{il}^4 + \sum_{i=1}^{m-1}c_i + \sum_{i=1}^{n}\frac{1}{2\varepsilon_{0i}^2}\left\|D_1 D_1^T\right\|^2\\
&+\left(\frac{1}{2\delta_1^2}\|D\|^4\|P\|^4 + \frac{1}{2\delta_2^2}\|D\|^4\|P\|^2|\lambda_{\max}\{P\}|^2\right)\|\bar{y}\|^4\\
&-\left(b\lambda_{min}\{P\} - 8\delta_1^2 b^2 s - 2\delta_2^2 b^2 r\right)\|\tilde{x}\|^4 - W_0\sum_{l-1}^{n} y_l^4\\
\leq &-\sum_{i=1}^{m-1}\sum_{l=1}^{n}\lambda_{il}z_{il}^4 + \sum_{i=1}^{m-1}c_i + \sum_{i=1}^{n}\frac{1}{2\varepsilon_{0i}^2}\left\|D_1 D_1^T\right\|^2\\
&-\left(b\lambda_{min}\{P\} - 8\delta_1^2 b^2 s - 2\delta_2^2 b^2 r\right)\|\tilde{x}\|^4\\
&-\left(\lambda_{\min}\{W_0\} - \frac{1}{2\delta_1^2}\|D\|^4\|P\|^4\right.\\
&\left.-\frac{1}{2\delta_2^2}\|D\|^4\|P\|^2|\lambda_{\max}\{P\}|^2\right)\|\bar{y}\|^4
\end{aligned}
\tag{34}
$$

Therefore, the closed-loop stochastic system is bounded in probability sense if the condition of the theorem holds. The proof is completed.

blacksquare

Remark 2: This Lyapunov function consists of three terms, where the first and the third therm use the 4-th power terms. These two terms are in line with the widely used Lyapunov function in formulating iterative backstepping control for stochastic systems because the 4-th powered term can handle the Itô correction term well; The second term is related to the state estimation error. Since the first and the third term can deal with the Itô correction, the second term can be selected as the lowest possible order.

IV. Parameter Optimisation

Based upon the structure of controller (27), the performance of the controller can be affected by two factors: the first virtual control input and the design parameters. The first virtual control input $\bar{\phi}_1(\bar{y})$ can be designed in the similar form to other virtual controls here

$$
\bar{\phi}_1(\bar{y}) = A_0\bar{y} - A_1\bar{y} - \tilde{\bar{x}}_2 - \frac{3}{4}\left[\varepsilon_{01}^2 y_1, \ldots, \varepsilon_{0n}^2 y_n\right]^T\|\bar{y}\|^4 \tag{35}
$$

Using this special form, Π_1 can be omitted in the structure of the controller because the Hessian matrix becomes zero. The stability of the closed-loop stochastic control system can be guaranteed naturally and the design parameters can be optimized to achieve the control objective, namely the stochastic coupling attenuation in the second moment sense.

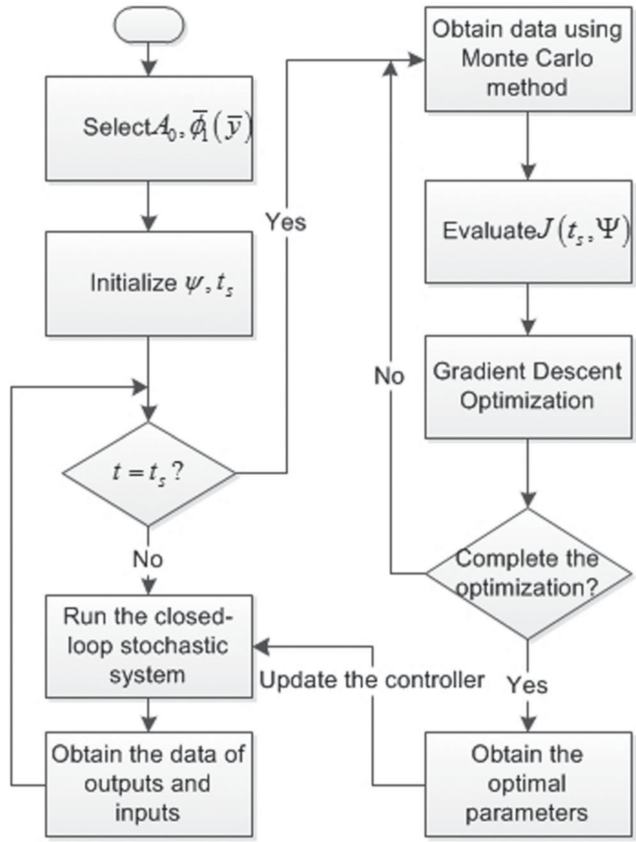

Fig. 1. The flow chart of the presented control algorithm.

Without loss of generality, it is assumed that parameters Λ_i and A_0 are pre-selected. The set of the tunable design parameter can be described as $\Psi := \{\varepsilon_{il} > 0 | i : 0 \to m, l : 1 \to n\}$, where Ψ can be rewritten as a column vector. To optimise these design parameters, the criterion (4) is extended as

$$
J(t_s, \Psi) = \min\left\{\Psi^T R\Psi + E\left\{\frac{1}{t_s}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^{t_s} \mathrm{w}_{ij}\Theta_{ij}^2(\Psi, t, y_i, y_j)\,dt\right\}\right\} \tag{36}
$$

where $w_{ij} = w_{ji}$ denotes the weight for each elements of the covariance matrix and t_s stands for the operating time of the closed-loop stochastic system $\Theta_{ij}(\Psi, t, y_i, y_j) := (y_i(\Psi, t) - E\{y_i\})(y_j(\Psi, t) - E\{y_j\})$. R is a real positive number.

To minimize this performance criterion, if R is selected sufficiently large, then we have $\frac{\partial^2 J(t_s, \Psi)}{\partial\Psi\partial\Psi} > 0$, which means that the gradient descent optimization can be used and the convergence can be guaranteed. Furthermore, the control inputs increase if the parameter $\Psi > 0$ increase, thus the initial value of the parameters can be selected reasonably.

Notice that different operating times t_s lead to the changes of parameters dynamically. For any arbitrary t_s, the value of the performance criterion (36) can be calculated and the optimal design parameters are obtained by Monte Carlo method [13] and

$$
\varepsilon_{ij}^{(k+1)} = \varepsilon_{ij}^{(k)} - \xi\nabla_{\varepsilon_{ij}^{(k)}} J\left(t_s, \Psi^{(k)}\right) \tag{37}
$$

where k denotes the iterative index and ξ is the step size.

In the end, the design algorithm in this section can be shown by the flow chart in Fig. 1.

Remark 3: The proposed approach in this section can be used with different $\bar{\phi}_1(\bar{y})$ for the closed-loop stochastic system if the conditions of Theorem 3 can be guaranteed.

V. An Illustrative Example

To illustrate the design procedure, a simple multi-variable bilinear stochastic model are given as follows:

$$\begin{aligned} d\bar{x}_1 &= \left(\begin{bmatrix} -1 & 0.5 \\ 0 & -2 \end{bmatrix} \bar{x}_1 + \bar{x}_2 \right) dt + \bar{x}_1 \begin{bmatrix} 0.5 & 1 \end{bmatrix} d\beta_t \\ d\bar{x}_2 &= \left(\begin{bmatrix} -1.5 & 0 \\ -0.5 & -1 \end{bmatrix} \bar{x}_2 + \bar{u} \right) dt + \bar{x}_1 \begin{bmatrix} -1 & 0 \end{bmatrix} d\beta_t \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{38}$$

where A_1, A_2, D_1 and D_2 have been given correspondingly.

The feedback gain matrices of the linear estimator can be chosen as $L_1 = L_2 = diag\{15, 15\}$. Therefore, the closed-loop stochastic system with estimator can be expressed by

$$\begin{aligned} d\begin{bmatrix} \tilde{\bar{x}}_1 \\ \tilde{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} -3 & 0.5 & 1 & 0 \\ 0 & -5 & 0 & 1 \\ -2 & 0 & -1.5 & 0 \\ 0 & -3 & -0.5 & -1 \end{bmatrix} \begin{bmatrix} \tilde{\bar{x}}_1 \\ \tilde{\bar{x}}_2 \end{bmatrix} dt \\ &\quad + \begin{bmatrix} \bar{y} & 0 \\ 0 & \bar{y} \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ -1 & 0 \end{bmatrix} d\beta_t \\ d\bar{y} &= \left(\begin{bmatrix} -1 & 0.5 \\ 0 & -2 \end{bmatrix} \bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2 \right) dt + \bar{y} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}^T d\beta_t \\ d\hat{\bar{x}}_2 &= \left(\begin{bmatrix} -1.5 & 0 \\ -0.5 & -1 \end{bmatrix} \hat{\bar{x}}_2 + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \tilde{\bar{x}}_1 + \bar{u} \right) dt \end{aligned} \tag{39}$$

where matrices $\tilde{A}$, Y and D are obtained with $\tilde{A}$ being Hurwitz.

As we discussed before, the first virtual control input can be chosen as

$$\begin{aligned} \bar{\phi}_1(\bar{y}) = &- \begin{bmatrix} 20 & 0 \\ 0 & 25 \end{bmatrix} \bar{y} - \begin{bmatrix} -1 & 0.5 \\ 0 & -2 \end{bmatrix} \bar{y} \\ &- \tilde{\bar{x}}_2 - \frac{3\|\bar{y}\|^4}{4} \begin{bmatrix} \varepsilon_{01}^2 & 0 \\ 0 & \varepsilon_{02}^2 \end{bmatrix} \bar{y} \end{aligned} \tag{40}$$

then the controller can be designed as

$$\begin{aligned} \bar{u} = &\Lambda_1 \bar{z}_1 + \Phi_1(\varepsilon_{01}, \varepsilon_{02}) \left(A_1 \bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2 \right) - F_2 \\ &+ \frac{1}{2} \Pi_1(\varepsilon_{01}, \varepsilon_{02}) - \frac{3}{4} \Xi_1(\varepsilon_{01}, \varepsilon_{02}, \varepsilon_{11}, \varepsilon_{12}) \|\bar{y}\|^4 \end{aligned} \tag{41}$$

The controller is affected by the design parameters ε_{ij} directly when we pre-select $\Lambda_1 = diag\{-1.5, -1\}$.

In the simulation, operating time has been selected as $t_s = 0.01$ s, and the curves of the performance of the closed-loop stochastic system are given in Figs. 2—5. The output trajectories are shown in Fig. 2 where the stochastic outputs are stabilized rapidly. The Fig. 3 depicts the control input signal. In Fig. 4, the values of the performance criterion are given as a smooth curve due to Monte Carlo method. It is shown that the value of J descends along with the search of the optimal design parameters. The estimation error of the closed-loop system is illustrated by Fig. 5 where it can be seen that all the values of the errors converge to 0.

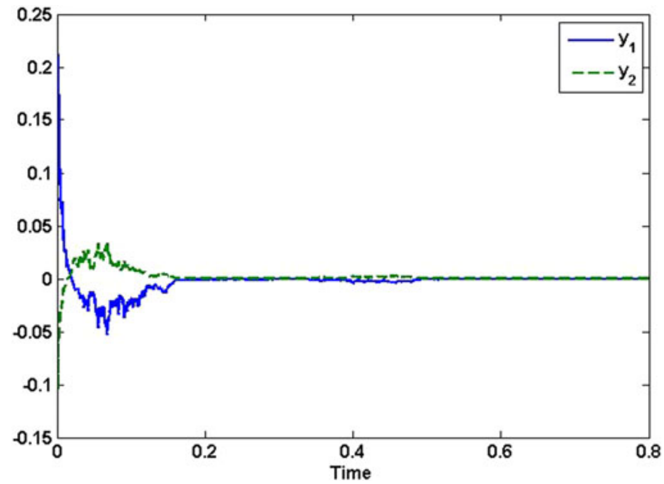

Fig. 2. Output trajectories of the closed-loop stochastic system.

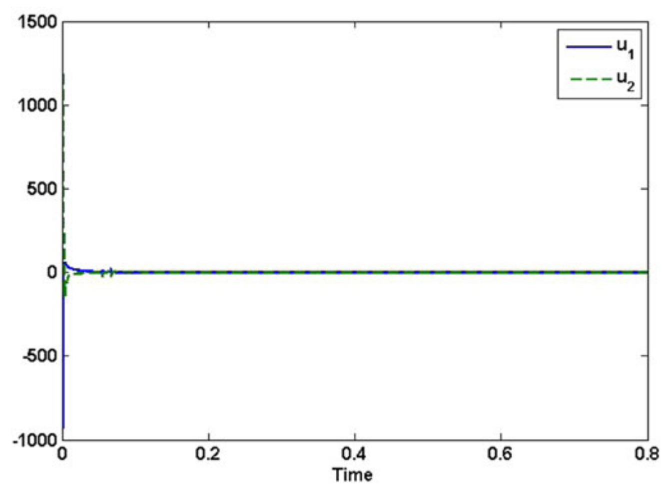

Fig. 3. The control input signal.

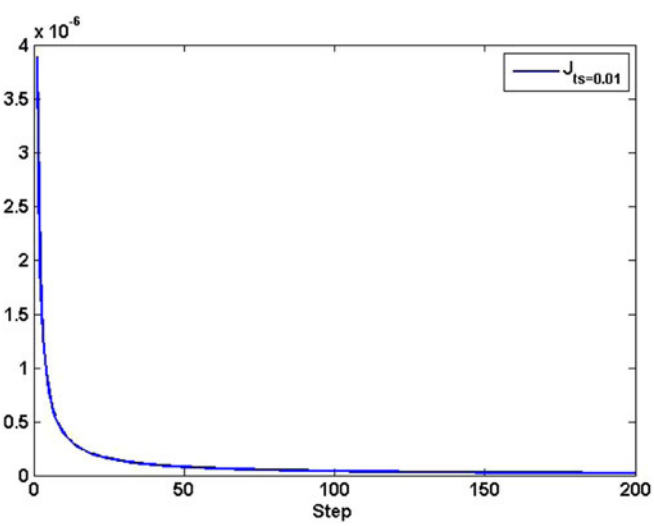

Fig. 4. The value of J for single t_s.

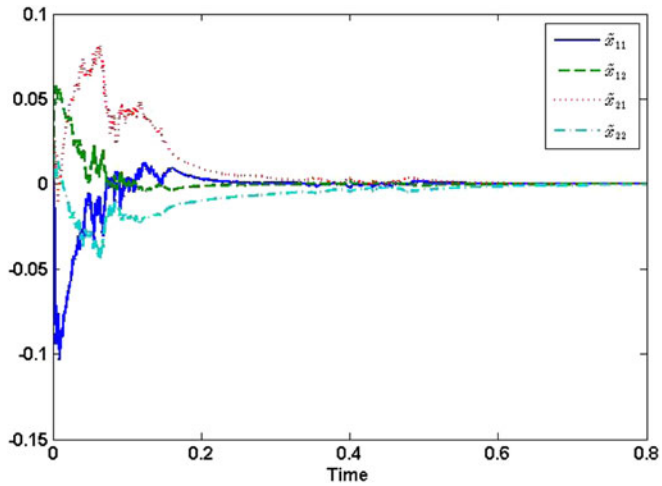

Fig. 5. The estimation error $\tilde{x}$.

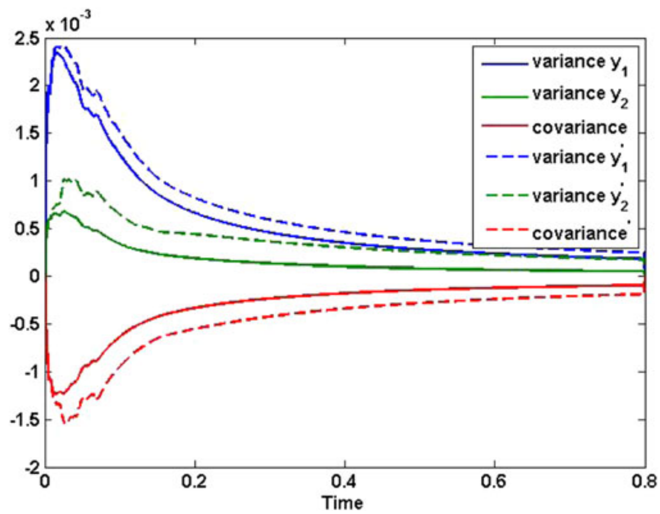

Fig. 6. The variance and covariance of the system outputs.

For this example, the optimal design parameters can be obtained as $\varepsilon_{01} = 3.6737$, $\varepsilon_{02} = 3.6008$, $\varepsilon_{11} = 3.3185$, $\varepsilon_{12} = 3.7384$. To compare the performance between the different design parameters, all the design parameters are set to 1. The result is given in Fig. 6 where the prime in legend denotes the results of the fixed parameters. It is shown that the performance of the controller with optimal design parameters has the covariance close to zeros and the output variances decrease simultaneously.

VI. Conclusion

For a class of multi-variable bilinear stochastic systems, the problem of stochastic coupling attenuation is discussed in this technical note based on output feedback stabilization, where the block backstepping design is used. Motivated by the concept of probability independence, stochastic decoupling is proposed as a new concept and the stochastic coupling attenuation in the second moment sense is defined based on pairwise independence which is described by covariance matrix. Therefore, the covariance matrix of the system outputs is considered as the control objective where the minimization of each elements of the covariance matrix is carried out via the control design. An output feedback block backstepping design is presented to stabilize the systems in probability sense. Meanwhile, a novel performance criterion is constructed using the definition of the stochastic coupling attenuation in the second moment sense and the well-known gradient descent optimization is used in the control design. By analyzing the results of the numerical example, it has been shown that the proposed control algorithm is effective and achievable.

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